

# On the $8n^2$ -inequality

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## Abstract

We give a complete proof of the so called  $8n^2$ -inequality, a local inequality for the self-intersection of a movable linear system at an isolated centre of a non canonical singularity. The inequality was suggested and several times published by I.Cheltsov but some of his arguments are faulty. We explain the mistake and replace the faulty piece by a correct argument.

**1. Introduction.** The aim of this note is to give a complete proof of the so called  $8n^2$ -inequality, correcting the mistakes in [1,2] and some other papers where the erroneous arguments were reproduced. That inequality makes it possible to prove birational (super)rigidity of several types of Fano varieties of anticanonical degrees 6,7 and 8. One step in the arguments of [1,2] is based on an erroneous claim which it is unclear how to correct (and whether that step can be corrected at all, following the approach of [1,2]).

In this note, we replace the faulty step by a different argument, thus making the proof complete.

The note is organized as follows. In Sec.2 we formulate the  $8n^2$ -inequality and reproduce briefly that part of the proof in [1,2], which is correct. In Sec. 3 we present the new arguments, completing the proof. In Sec. 4 we discuss the mistakes in [1,2], one of which is really serious and undermines the whole proof. The list of references given in Sec. 5 is far from being complete: the faulty arguments were published in a few more papers.

It is worth mentioning that in the survey [3] the arguments are the same as those of [1,2], with one exception: the very claim, for which [1,2] give the faulty arguments, is presented in [3] with essentially no proof at all.

**2. The inequality and start of the proof.** Let  $o \in X$  be a germ of a smooth variety of dimension  $\dim X \geq 4$ . Let  $\Sigma$  be a movable system on  $X$  and the effective cycle

$$Z = (D_1 \circ D_2),$$

where  $D_1, D_2 \in \Sigma$  are generic divisors, its self intersection. Let us blow up the point  $o$  on  $X$ :

$$\varphi: X^+ \rightarrow X,$$

$E = \varphi^{-1}(o) \cong \mathbb{P}^{\dim X - 1}$  is the exceptional divisor. The strict transforms of the system  $\Sigma$  and the cycle  $Z$  on  $X^+$  denote by the symbols  $\Sigma^+$  and  $Z^+$ , respectively.

**Theorem ( $8n^2$ -inequality).** *Assume that the pair*

$$(X, \frac{1}{n}\Sigma)$$

*is not canonical, but canonical outside the point  $o$ , where  $n$  is some positive integer. There exists a linear subspace  $P \subset E$  of codimension two (with respect to  $E$ ) such, that the following inequality holds:*

$$\text{mult}_o Z + \text{mult}_P Z^+ > 8n^2.$$

**Proof.** Note that if  $\text{mult}_o Z > 8n^2$ , then for  $P$  we can take any subspace of codimension two in  $E$ .

We start the proof arguing as in [1,2].

Restricting  $\Sigma$  onto a germ of a generic smooth subvariety, containing the point  $o$ , we may assume that  $\dim X = 4$ . Moreover, we may assume that  $\nu = \text{mult}_o Z \leq 2\sqrt{2}n < 3n$ , since otherwise

$$\text{mult}_o Z \geq \nu^2 > 8n^2$$

and there is nothing to prove.

**Lemma 1.** *The pair*

$$(X^+, \frac{1}{n}\Sigma^+ + \frac{(\nu - 2n)}{n}E) \tag{1}$$

*is not log canonical, and the centre of any of its non log canonical singularities is contained in the exceptional divisor  $E$ .*

**Proof.** Let  $\lambda: \tilde{X} \rightarrow X$  be a resolution of singularities of the pair  $(X, \frac{1}{n}\Sigma)$  and  $E^* \subset \tilde{X}$  a prime exceptional divisor, realizing a non-canonical singularity of that pair. Then  $\lambda(E^*) = o$  and the Noether-Fano inequality holds:

$$\nu_{E^*}(\Sigma) > na(E^*).$$

For a generic divisor  $D \in \Sigma$  we get  $\varphi^*D = D + \nu E$ , so that

$$\nu_{E^*}(\Sigma) = \nu_{E^*}(\Sigma^+) + \nu \cdot \nu_{E^*}(E)$$

and

$$a(E^*, X) = a(E^*, X^+) + 3\nu_{E^*}(E).$$

From here we get

$$\begin{aligned} \nu_{E^*} \left( \frac{1}{n}\Sigma^+ + \frac{\nu - 2n}{n}E \right) &= \nu_{E^*} \left( \frac{1}{n}\Sigma \right) - 2\nu_{E^*}(E) > \\ &> a(E^*, X^+) + \nu_{E^*}(E) \geq a(E^*, X^+) + 1, \end{aligned}$$

which proves the lemma.

Let  $R \ni o$  be a generic three-dimensional germ,  $R^+ \subset X^+$  its strict transform on the blow up of the point  $o$ . For a small  $\varepsilon > 0$  the pair

$$\left( X^+, \frac{1}{1+\varepsilon} \frac{1}{n} \Sigma^+ + \frac{\nu-2n}{n} E + R^+ \right)$$

still satisfies the connectedness principle (with respect to the morphism  $\varphi: X^+ \rightarrow X$ ), so that the set of centres of non log canonical singularities of that pair is connected. Since  $R^+$  is a non log canonical singularity itself, we obtain, that there is a non log canonical singularity of the pair (1), the centre of which on  $X^+$  is of positive dimension, since it intersects  $R^+$ .

Let  $Y \subset E$  be a centre of a non log canonical singularity of the pair (1) that has the maximal dimension.

If  $\dim Y = 2$ , then consider a generic two-dimensional germ  $S$ , intersecting  $Y$  transversally at a point of general position. The restriction of the pair (1) onto  $S$  is not log canonical at that point, so that, arguing as in [1,2], we see that

$$\text{mult}_Y(D_1^+ \circ D_2^+) > 4 \left( 3 - \frac{\nu}{n} \right) n^2,$$

so that

$$\begin{aligned} \text{mult}_o Z &\geq \nu^2 + \text{mult}_Y(D_1^+ \circ D_2^+) \deg Y > \\ &> (\nu - 2n)^2 + 8n^2, \end{aligned}$$

which is what we need.

If  $\dim Y = 1$ , then, since the pair

$$\left( R^+, \frac{1}{1+\varepsilon} \frac{1}{n} \Sigma_R^+ + \frac{\nu-2n}{n} E_R \right), \quad (2)$$

where  $\Sigma_R^+ = \Sigma^+|_{R^+}$  and  $E_R = E|_{R^+}$ , satisfies the condition of the connectedness principle and  $R^+$  intersects  $Y$  at  $\deg Y$  distinct points, we conclude that  $Y \subset E$  is a line in  $\mathbb{P}^3$ .

Now we need to distinguish between the following two cases: when  $\nu \geq 2n$  and when  $\nu < 2n$ . The methods of proving the  $8n^2$ -inequality in these two cases are absolutely different. Consider first the case  $\nu \geq 2n$ .

Let us choose as  $R \ni o$  a generic three-dimensional germ, satisfying the condition  $R^+ \supset Y$ . Since the pair (2) is effective (recall that  $\nu \geq 2n$ ), one may apply inversion of adjunction [4, Chapter 17] and conclude that the pair (2) is not log canonical at  $Y$ .

Now arguing in the same way as for  $\dim Y = 2$ , with  $R^+ \supset Y$ , we get the inequality

$$\text{mult}_Y(D_1^+|_{R^+} \circ D_2^+|_{R^+}) > 4 \left( 3 - \frac{\nu}{n} \right) n^2.$$

On the left in brackets we have the self-intersection of the movable system  $\Sigma_R^+$ , which breaks into two natural components:

$$(D_1^+|_{R^+} \circ D_2^+|_{R^+}) = Z_R^+ + Z_R^{(1)},$$

where  $Z_R^+$  is the strict transform of the cycle  $Z_R = Z|_R$  on  $R^+$  and the support of the cycle  $Z_R^{(1)}$  is contained in  $E_R$ . The line  $Y$  is a component of the effective cycle  $Z_R^{(1)}$ .

On the other hand, for the self-intersection of the movable linear system  $\Sigma^+$  we get

$$(D_1^+ \circ D_2^+) = Z^+ + Z_1,$$

where the support of the cycle  $Z_1$  is contained in  $E$ . From the genericity of  $R$  it follows that outside the line  $Y$  the cycles  $Z_R^{(1)}$  and  $Z_1|_{R^+}$  coincide, whereas for  $Y$  we get the equality

$$\text{mult}_Y Z_R^{(1)} = \text{mult}_Y Z^+ + \text{mult}_Y Z_1.$$

However,  $\text{mult}_Y Z_1 \leq \deg Z_1$ , so that

$$\begin{aligned} \text{mult}_o Z + \text{mult}_Y Z^+ &= \\ &= \nu^2 + \deg Z_1 + \text{mult}_Y Z^+ \geq \\ &\geq \nu^2 + \text{mult}_Y Z_R^{(1)} > 8n^2, \end{aligned}$$

which is what we need. This completes the case  $\nu \geq 2n$ .

Note that the key point in this argument is that the pair (2) is effective. For  $\nu < 2n$  inversion of adjunction can not be applied (as it was done in [3]). The additional arguments in [1,2], proving inversion of adjunction specially for this pair for  $\nu < 2n$ , are faulty.

**3. The technique of counting multiplicities.** Starting from this moment, assume that  $\nu < 2n$ .

Consider again the pair (2) for a generic germ  $R \ni o$ . Let  $y = Y \cap R^+$  be the point of (transversal) intersection of the line  $Y$  and the variety  $R^+$ . Since  $a(E_R, R) = 2$ , the non log canonicity of the pair (2) at the point  $y$  implies the non log canonicity of the pair

$$\left(R, \frac{1}{n} \Sigma_R\right)$$

at the point  $o$ , whereas the centre of some non log canonical (that is, log maximal) singularity on  $R^+$  is a point  $y$ .

Now the  $8n^2$ -inequality comes from the following fact.

**Lemma 2.** *The following inequality holds:*

$$\text{mult}_o Z_R + \text{mult}_y Z_R^+ > 8n^2,$$

where  $Z_R$  is the self-intersection of a movable linear system  $\Sigma_R$  and  $Z_R^+$  is its strict transform on  $R^+$ .

**Proof.** Consider the resolution of the maximal singularity of the system  $\Sigma_R$ , the centre of which on  $R^+$  is the point  $y$ :

$$\begin{array}{ccc} R_i & \xrightarrow{\psi_i} & R_{i-1} \\ \cup & & \cup \\ E_i & & B_{i-1}, \end{array}$$

where  $B_{i-1}$  is the centre of the singularity on  $R_{i-1}$ ,  $E_i = \psi_i^{-1}(B_{i-1})$  is the exceptional divisor,  $B_0 = o$ ,  $B_1 = y \in E_1$ ,  $i = 1, \dots, N$ , where the first  $L$  blow ups correspond to points, for  $i \geq L + 1$  curves are blown up. Since

$$\text{mult}_o \Sigma_R = \text{mult}_o \Sigma < 2n,$$

we get  $L < N$ ,  $B_L \subset E_L \cong \mathbb{P}^2$  is a line and for  $i \geq L + 1$

$$\deg[\psi_i|_{B_i}: B_i \rightarrow B_{i-1}] = 1,$$

that is,  $B_i \subset E_i$  is a section of the ruled surface  $E_i$ .

Consider the graph of the sequence of blow ups  $\psi_i$ .

**Lemma 3.** *The vertices  $L + 1$  and  $L - 1$  are not connected by an arrow:*

$$L + 1 \nrightarrow L - 1.$$

**Proof.** Assume the converse:  $L + 1 \rightarrow L - 1$ . This means that

$$B_L = E_L \cap E_{L-1}^L$$

is the exceptional line on the surface  $E_{L-1}^L$  and the map

$$E_{L-1}^{L+1} \rightarrow E_{L-1}^L$$

is an isomorphism. As usual, set

$$\nu_i = \text{mult}_{B_{i-1}} \Sigma_R^{i-1},$$

$i = 1, \dots, N$ . Let us restrict the movable linear system  $\Sigma_R^{L+1}$  onto the surface  $E_{L-1}^{L+1}$  (that is, onto the plane  $E_{L-1} \cong \mathbb{P}^2$  with the blown up point  $B_{L-1}$ ). We obtain a non-empty (but, of course, not necessarily movable) linear system, which is a subsystem of the complete linear system

$$|\nu_{L-1}(-E_{L-1}|_{E_{L-1}}) - (\nu_L + \nu_{L+1})B_L|.$$

Since  $(-E_{L-1}|_{E_{L-1}})$  is the class of a line on the plane  $E_{L-1}$ , this implies that

$$\nu_{L-1} \geq \nu_L + \nu_{L+1} > 2n,$$

so that the more so  $\nu_1 = \nu > 2n$ . A contradiction. Q.E.D. for the lemma.

Set, as usual,

$$m_i = \text{mult}_{B_{i-1}} (Z_R)^{i-1},$$

$i = 1, \dots, L$ , so that, in particular,

$$m_1 = \text{mult}_o Z_R \quad \text{and} \quad m_2 = \text{mult}_y Z_R^+.$$

Let  $p_i \geq 1$  be the number of paths in the graph of the sequence of blow ups  $\psi_i$  from the vertex  $N$  to the vertex  $i$ , and  $p_N = 1$  by definition, see [5,6]. By what we proved,

$$p_N = p_{N-1} = \dots = p_L = p_{L-1} = 1,$$

and the number of paths  $p_i$  for  $i \leq L$  is the number of paths from the vertex  $L$  to the vertex  $i$ . By the technique of counting multiplicities [5,6], we get the inequality

$$\sum_{i=1}^L p_i m_i \geq \sum_{i=1}^N p_i \nu_i^2$$

and, besides, the Noether-Fano inequality holds:

$$\sum_{i=1}^N p_i \nu_i > n \left( 2 \sum_{i=1}^L p_i + \sum_{i=L+1}^N p_i \right).$$

(In fact, a somewhat stronger inequality holds, the *log* Noether-Fano inequality, but we do not need that.) From the last two estimates one obtains in the standard way [5,6] the inequality

$$\sum_{i=1}^L p_i m_i > \frac{(2\Sigma_0 + \Sigma_1)^2}{\Sigma_0 + \Sigma_1} n^2,$$

where  $\Sigma_0 = \sum_{i=1}^L p_i$  and  $\Sigma_1 = \sum_{i=L+1}^N p_i = N - L$ . Taking into account that for  $i \geq 2$  we get

$$m_i \leq m_2$$

and the obvious inequality  $(2\Sigma_0 + \Sigma_1)^2 > 4\Sigma_0(\Sigma_0 + \Sigma_1)$ , we obtain the following estimate

$$p_1 m_1 + (\Sigma_0 - p_1) m_2 > 4n^2 \Sigma_0.$$

Now assume that the claim of the lemma is false:

$$m_1 + m_2 \leq 8n^2.$$

**Lemma 4.** *The following inequality holds:  $\Sigma_0 \geq 2p_1$ .*

**Proof.** By definition,

$$p_1 = \sum_{i \rightarrow 1} p_i,$$

however, by Lemma 3 from  $i \rightarrow 1$  it follows that  $i \leq L$ , so that  $p_1 \leq \Sigma_0 - p_1$ , which is what we need. Q.E.D. for the lemma.

Now, taking into account that  $m_2 \leq m_1$ , we obtain

$$\begin{aligned} p_1 m_1 + (\Sigma_0 - p_1) m_2 &= p_1(m_1 + m_2) + (\Sigma_0 - 2p_1) m_2 \leq \\ &\leq 8p_1 n^2 + (\Sigma_0 - 2p_1) \cdot 4n^2 = 4n^2 \Sigma_0. \end{aligned}$$

This is a contradiction. Q.E.D. for Lemma 2.

Proof of our theorem is complete.

**4. On the faulty arguments.** The proof of the  $8n^2$ -inequality given in [1] is invalid. We refer the reader to that paper (the numbers of pages and claims correspond to the archive version given in the reference [1]).

**Mistake 1.** Page 6 in the archive version of [1], just after the proof of Lemma 27. The claim that the intersection of the divisor  $S$  with the curve  $C$  is either trivial or consists of more than one point, is wrong. The divisor  $F$  contains a 3-dimensional family of smooth rational curves, intersecting transversally a generic divisor  $S$  at one point. Namely, the surface

$$\bar{E} \cap F$$

(in  $E \cong \mathbb{P}^3$  we blow up the line  $L$ , and  $\bar{E} \cap F$  is the exceptional divisor) is  $\mathbb{P}^1 \times \mathbb{P}^1$ , so that any curve  $C$  of bidegree  $(1,1)$  can be used. (This error can be corrected, for instance, by restricting the linear system onto the exceptional divisor  $E$  and showing that in the case of a  $(1,1)$  curve  $\nu > 2n$ , so that the arguments based on inversion of adjunction work.)

**Mistake 2.** Page 5 of [1]: Corollary 24 is not true. The intersection  $S \cap C$  can well be empty.

Indeed, from the following two facts:

**A.** the set  $LCS(S, (B^W + \bar{E} + 2F)|_S)$  either consists of one point or contains a curve (which is deduced from the connectedness principle (Theorem 14)), and

**B.** there is a curve  $C$ , a section of the bundle  $F \rightarrow L$ , such that  $C$  is the unique element of the set  $LCS(W, B^W + \bar{E} + aF)$ ,  $a = 1, 2$ , which is contained in  $F$  and dominates  $L$  (just above, page 5),

it **does not** follow that  $LCS(S, (B^W + \bar{E} + 2F)|_S)$  is the intersection  $S \cap C$  (and this is exactly how Corollary 24 is proved), since  $LCS(S, (B^W + \bar{E} + 2F)|_S)$  can well be the intersection of  $S$  with the centre of a non log canonical singularity of the pair  $(W, B^W + \bar{E} + 2F)$ , which **does not** dominate  $L$ : for instance, with a line in the fiber of the bundle  $F \rightarrow L$  (this is a  $\mathbb{P}^2$ -bundle over  $L$ ). In particular, if the set  $LCS(W, B^W + \bar{E} + aF)$ ,  $a = 1, 2$ , is a connected union of two curves:

(1) a line in a fiber  $F \rightarrow L$  and

(2) a curve of bidegree  $(1,0)$  on the surface  $\bar{E} \cap F$ , which is  $\mathbb{P}^1 \times \mathbb{P}^1$ , that is, a section of  $F \rightarrow L$ , contained in  $\bar{E} \cap F$  and having the zero self-intersection on that surface,

then A and B hold, but there is no contradiction at all.

The “proof” of Corollary 24 is faulty, because the (correct) claim “the centre of any singularity, *dominating*  $L$ , is  $C$ ” is used actually as the claim “the centre of any singularity is  $C$ ”. The example above shows that the arguments of [1] are faulty and give no proof of the  $8n^2$ -inequality. (In [2] the same arguments are given as those used in [1] for proving B, after which it is claimed that  $C$  is the unique element of the set  $LCS(\dots)$ , without mentioning that  $L$  is dominated. Here it is easier to see the point of trouble.)

## 5. References.

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